

On a class of third order mappings with two rational invariants

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Abstract

A novel family of integrable third order maps is presented. Each map possesses, by construction, a pair of rational invariants and a commuting map from the same class. The 3-dimensional invariant curve is parametrized, in general, by an elliptic curve.

1 Introduction

The construction of many nontrivial rational mappings is based on the trivial idea: if one root of a quadratic equation is known then the second one is found in rational form. The most important example is the family of QRT mappings [1] introduced as follows: let $f(x, y)$, $g(x, y)$ be biquadratic polynomials and $I = f/g$, then the corresponding map $(x, y) \rightarrow (\tilde{x}, \tilde{y})$ is defined by equations

$$I(x, y) = I(\tilde{x}, y) = I(\tilde{x}, \tilde{y})$$

where the solutions $\tilde{x} = x$, $\tilde{y} = y$ are ignored. Obviously, the resulting map is the composition of two rational involutions and I is its invariant by construction.

This can be generalized in several ways. For example, one may consider the rational mapping $(x, y, z) \rightarrow (\tilde{x}, \tilde{y}, \tilde{z})$ defined by equations

$$I(x, y, z) = I(\tilde{x}, y, z) = I(\tilde{x}, \tilde{y}, z) = I(\tilde{x}, \tilde{y}, \tilde{z})$$

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where $I = f/g$ is ratio of two three-quadratic polynomials. The generalization for any number of variables is straightforward. In contrast to the QRT case such mappings are not integrable in general. However, in papers [2, 3] some instances were found when a second polynomial or rational invariant exists. Several other examples of third order integrable mappings were studied in [4, 5].

In this paper we consider another possibility, assuming from the beginning that the map under construction possesses two invariants I, J , but on each step two of three variables are changed, rather than one. The idea of differentiating two invariants simultaneously was used previously in paper [6], but in a different manner. It is easy to see that if I and J are ratios of affine-linear polynomials on x, y, z (that is, numerators and denominators are of the first degree on each argument) then the resulting map is a composition of rational involutions again. Three basic involutions satisfy a simple identity and as a result one obtains two commuting maps which share the common invariants and generate the kagome lattice. Recall that the existence of commuting partner is a typical feature of integrable maps [7]. The generalization for any number of variables is also straightforward, however the resulting maps are not in general integrable.

It should be stressed that the presented family in no way covers all integrable cases of third order maps. Actually, it seems to be not so thick as the QRT-like maps from the papers cited above. On the other hand, the construction scheme seems to be more explicit. Of course, there should be intersections between these families, but the absence of any classification results makes the detailed comparison impossible for the present. Anyway, we see again that the most trivial ideas work more than once (cf. also the recent construction of Yang-Baxter maps [8]).

2 The triad mapping

Let f, g, h, k be affine-linear polynomials on x, y, z and $I = f/g, J = h/k$. Consider equations for the unknowns \tilde{y}, \tilde{z} :

$$I(x, y, z) = I(x, \tilde{y}, \tilde{z}), \quad J(x, y, z) = J(x, \tilde{y}, \tilde{z}).$$

This is a system of the form

$$a\tilde{y}\tilde{z} + b\tilde{y} + c\tilde{z} + d = 0, \quad A\tilde{y}\tilde{z} + B\tilde{y} + C\tilde{z} + D = 0$$

with coefficients depending rationally on x, y, z . Obviously, it is equivalent to a quadratic equation and since the solution $(\tilde{y}, \tilde{z}) = (y, z)$ is known, hence

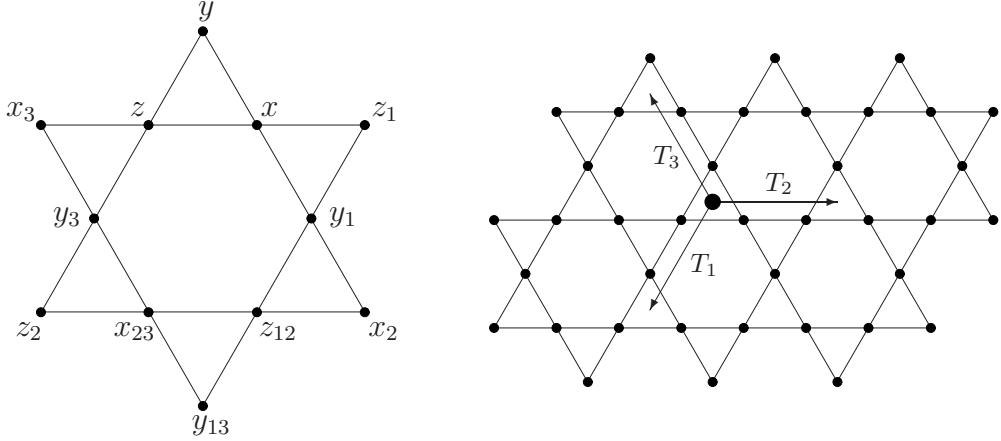


Fig. 1: Graphical representation of the identities (1), (2) and the kagome lattice

the second solution can be easily found in rational form. This defines the map $(x, y, z) \mapsto (x, \tilde{y}, \tilde{z})$. Changing the roles of the variables we obtain three rational mappings defined by equations

$$\begin{aligned} R_1 : \quad & I(x, y, z) = I(x, y_1, z_1), \quad J(x, y, z) = J(x, y_1, z_1), \\ R_2 : \quad & I(x, y, z) = I(x_2, y, z_2), \quad J(x, y, z) = J(x_2, y, z_2), \\ R_3 : \quad & I(x, y, z) = I(x_3, y_3, z), \quad J(x, y, z) = J(x_3, y_3, z) \end{aligned}$$

assuming that the identical solutions are always ignored. By construction, these maps are involutive:

$$R_1^2 = R_2^2 = R_3^2 = \text{id}. \quad (1)$$

The following property is far from being obvious.

Theorem 1. *Maps R_i satisfy the identity*

$$R_1 R_2 R_3 = R_3 R_2 R_1. \quad (2)$$

Proof. We present the computational proof based on the fact that all points lie on the invariant curve $f = Ig, h = Jk$ which is the intersection of two surfaces of the form

$$A : \quad a_1XYZ + a_2XY + a_3XZ + a_4YZ + a_5X + a_6Y + a_7Z + a_8 = 0. \quad (3)$$

Consider five points

$$(x_2, y_1, z_{12}) \xleftarrow{R_2} (x, y_1, z_1) \xleftarrow{R_1} (x, y, z) \xrightarrow{R_3} (x_3, y_3, z) \xrightarrow{R_2} (x_{23}, y_3, z_2) \quad (4)$$

(the enumeration is shown on the fig. 1). It is sufficient to prove that the values of y_{13} obtained in two different ways coincide. This is equivalent to the statement that the invariant curve intersects the straight line $L : (X, Z) = (x_{23}, z_{12})$.

Let $L \cap A = (x_{23}, y^*, z_{12})$. Consider the intersection lines of the surface (3) with the planes $X = x$, $Y = y_3$ and $Z = z_{12}$. Let (ξ, y_3, z_{12}) , (x, η, z_{12}) and (x, y_3, ζ) be the mutually common points of these lines. Then the following equations holds:

$$\begin{aligned} X = x &: \det([y, z], [y_1, z_1], [\eta, z_{12}], [y_3, \zeta]) = 0, \\ Y = y_3 &: \det([x_3, z], [x_{23}, z_2], [x, \zeta], [\xi, z_{12}]) = 0, \\ Z = z_{12} &: \det([x_2, y_1], [x_{23}, y^*], [\xi, y_3], [x, \eta]) = 0 \end{aligned}$$

where the notation $[p, q] = (pq, p, q, 1)^T$ is used.

Now we find y^* from the last equation where ξ and η are eliminated by use of the first and second ones. The remarkable fact, proved by direct and rather tedious computation is that ζ cancels out and y^* does not depend on it. This means that any surface of the form (3) passing through five points (4) passes also through the point (x_{23}, y^*, z_{12}) . Since the invariant curve is the intersection of two such surfaces, it also runs through this point and $y_{13} = y^*$. \square

As a corollary we immediately obtain that the mappings

$$T_1 = R_2 R_3, \quad T_2 = R_3 R_1, \quad T_3 = R_1 R_2$$

satisfy the identities

$$T_i T_j = T_j T_i, \quad T_1 T_2 T_3 = \text{id}.$$

Therefore we have, in general, a pair of commuting mappings which generate the kagome lattice. However, for some special choices of I, J this lattice may be reduced due to additional identities for the generators, see Examples 3, 4.

The projection of the invariant curve onto the coordinate plane (x, y) is defined by the equation $b(x, y) = F \partial_z H - H \partial_z F = 0$ where $F = f - Ig$, $H = h - Jk$. Generically, this is a biquadratic curve of genus 1, and the invariant curve is parametrized by the point on the elliptic curve $X^2 = r(x) = (\partial_y b)^2 - 2b \partial_y^2 b$.

3 Examples

Here we consider only few very particular examples of the presented mappings. The investigation of the whole family is probably an interesting but

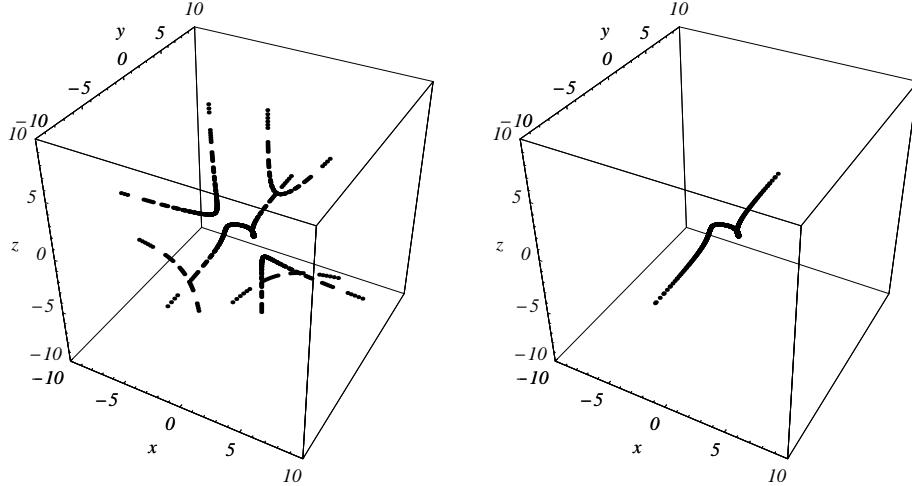


Fig. 2: Ex. 1. The iterations of T_1 and T_2 .

also a very difficult problem. The size of this family can be estimated roughly as follows. An affine-linear polynomial on x, y, z contains 2^3 coefficient parameters. In the ratios I, J two parameters are scaled out and we also have to take into account the 3-parametric group of Möbius transformations which acts on each variable independently as well as on the invariants. Therefore, the total number of essential parameters in the mappings under consideration is at most $4 \cdot 2^3 - 2 - (2 + 3) \cdot 3 = 15$ (the analogous reasoning gives $2 \cdot 3^2 - 1 - (1 + 2) \cdot 3 = 8$ for the QRT mappings and $2 \cdot 3^3 - 1 - (1 + 3) \cdot 3 = 41$ for its 3-component generalization mentioned in Introduction).

Example 1. A generic mapping. The simplest way to get some experience is to generate maps for the random choice of the coefficients in I, J and to iterate the random initial data. It turns out that already coefficients with the random values $0, \pm 1$ provide, as a rule, the nondegenerate map. The fig. 2 plots the images of the point $(x, y, z) = (1/2, 1/2, -1)$ under the mappings T_1 and T_2 for the invariants

$$I = \frac{y + z + xy - xyz}{x - z + xy}, \quad J = \frac{1 + x - z - xy - xz - yz - xyz}{1 - x + y - z - xy - xz + yz - xyz}.$$

Here we see that T_2 runs through only one branch of the invariant curve. Although the invariants look not too complicated, the corresponding maps are extremely bulky. For example, three components of R_1 contains in total 84 terms and T_1 contains 831 terms.

Example 2. A mapping with polynomial invariants. As in QRT case, more simple, but still nontrivial maps can be obtained already for polynomial

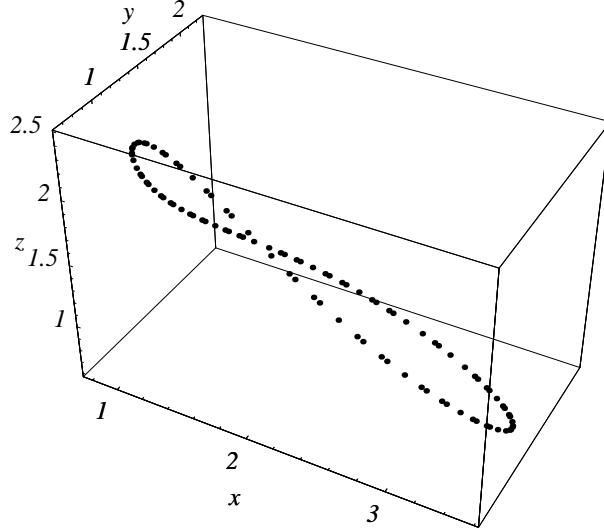


Fig. 3: Ex. 2. Iterations of the mapping T_1 , $a = 3, b = 1, c = 2$; initial values $(x, y, z) = (1, 2, 1)$

invariants. Let

$$I = x + y + z - xyz, \quad J = z(x + ay) + bx + cy$$

then the corresponding involutions are

$$R_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ \frac{x^2 z - x + az + c}{x^2 + a} \\ \frac{cxy + x + ay - c}{x^2 + a} \end{pmatrix}, \quad R_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{ay^2 z - ay + z + b}{by + 1} \\ y \\ \frac{bxy + x + ay - b}{ay^2 + 1} \end{pmatrix},$$

$$R_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{ayz^2 + cyz + (1 - a)z + b - c}{z(z + b)} \\ \frac{xz^2 + bxz + (a - 1)z + c - b}{z(az + c)} \\ z \end{pmatrix}$$

The iterations of the map $T_1 = R_2 R_3$ are shown on the fig. 3.

Example 3. Reduced group. In some cases the involutions R_i may satisfy additional identities. For example, this happens if invariants are symmetric with respect to a pair of variables. Consider the invariants:

$$I = xy + z, \quad J = (x + y)z.$$

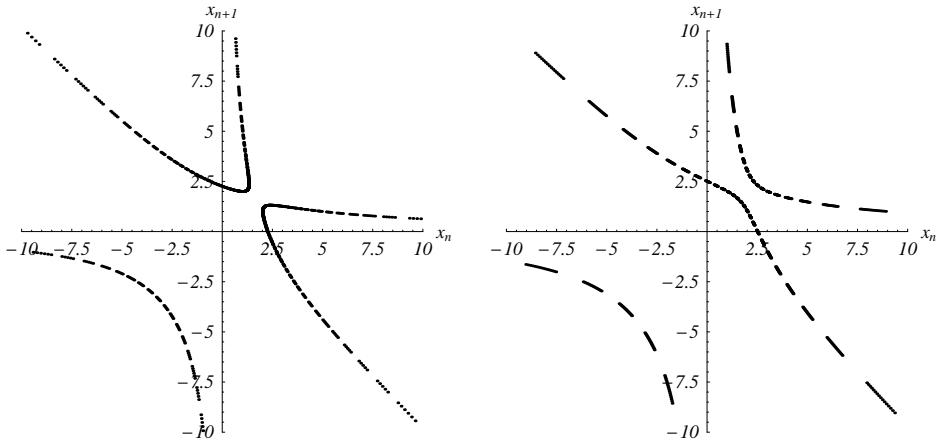


Fig. 4: Ex. 3. The iterations of the various initial data

One can check straightforwardly that the corresponding maps

$$R_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z/x - x \\ x(x+y) \end{pmatrix}, \quad R_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z/y - y \\ y \\ y(x+y) \end{pmatrix}, \quad R_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ z \end{pmatrix}$$

satisfy, in addition to the identities (1), (2), the relation $R_2 R_3 = R_3 R_1$. This means that in this case we obtain only one mapping $T_1 = T_2$ while $T_3 = T_1^{-2}$. It should be noted that such sort of group reduction is not related to the polynomiality of the invariants or to the degeneration of the invariant curve. Indeed, in this example its projection on (x, y) plane is the curve $(xy - I)(x + y) + J = 0$ of genus 1 iff $J \neq 0$.

The map T_1 generates the discrete system

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{n+1} = \begin{pmatrix} z/x - x \\ x \\ x(x+y) \end{pmatrix}_n$$

which can be easily rewritten as the third-order difference equation (see fig. 4)

$$(x_{n+3} + x_{n+2})x_{n+2} = x_{n+1}(x_{n+1} + x_n).$$

In this notation the invariants take the form

$$I = x_n(x_{n+1} + x_n + x_{n-1}), \quad J = (x_{n+1} + x_n)x_n(x_n + x_{n-1}).$$

Example 4. Finite group. Even more degeneracy occurs when all involutions R_i commute and generate only a few points on the invariant curve.

This happens, for example, if invariants are symmetric with respect to all variables. Obviously, in this case R_i are just permutations. Not so trivial example is given by invariants

$$I = xy + z, \quad J = \frac{yz + x}{xz + y}.$$

Here the involutions

$$R_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ \frac{xyz + x^2 + z^2 - 1}{xz + y} \\ \frac{xz + y}{x - x^3 + xy^2 + yz} \end{pmatrix}, \quad R_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{xyz + y^2 + z^2 - 1}{yz + x} \\ y \\ \frac{y - y^3 + yx^2 + xz}{yz + x} \end{pmatrix},$$

$$R_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ z \end{pmatrix}$$

also turn out to be commutative.

Example 5. One involution is identical. Consider the invariants

$$I = xy + z, \quad J = \frac{xz}{y + z}.$$

It is easy to check that the system for the map R_1 is equivalent to

$$y_1 z = y z_1, \quad x(y_1 - y) = z - z_1$$

and has only identical solution. Therefore in this case R_1 is actually absent. However, the rest involutions still generate the nontrivial mapping

$$T_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z(x - 1) \\ y + z \\ -y - z \\ x(y + z) \end{pmatrix}.$$

Its iterations are shown on the fig. 5. Note that the invariant curve is rational:

$$y = \frac{I(x - J)}{x^2 - Jx + J}, \quad z = \frac{IJ}{x^2 - Jx + J}.$$

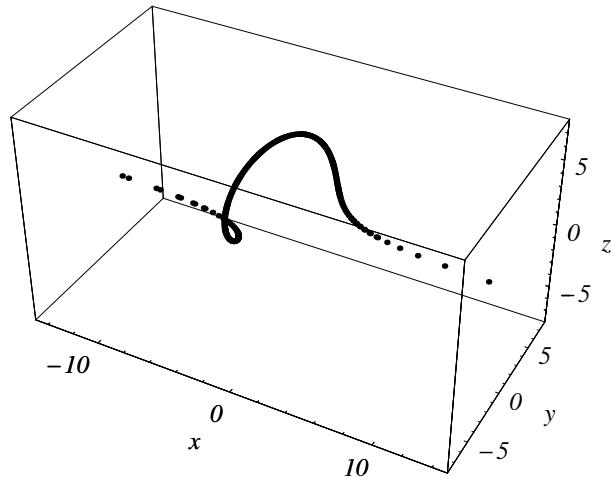


Fig. 5: Ex. 5. The mapping T_1 .

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